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Note

Density and power graphs in graph homomorphism problem☆

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Abstract

We introduce two necessary conditions for the existence of graph homomorphisms based on the concepts of density and power graph. As corollaries, we obtain a lower bound for the fractional chromatic number, and we set forward elementary proofs of the facts that the circular chromatic number of the Petersen graph is equal to three and the fact that the Coxeter graph is a core.

Keywords: Graph homomorphism; Power graph; Density; Pentagon conjecture

1. Introduction

Throughout the article we only consider finite graphs and refer the reader to [7,17] for notations. A *homomorphism* $f: G \longrightarrow H$ from a graph G to a graph H is a map $f: V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. Moreover, $\mathrm{Hom}[G, H]$ denote the set of homomorphisms from G to H (for more on graphs homomorphisms see [2,3,5,7]).

If n and d are positive integers with $n \geqslant 2d$, then the *circular complete graph* $K_{(n,d)}$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ in which v_i is connected to v_j if and only if $d \leqslant |i-j| \leqslant n-d$. A graph G is said to be (n, d)-colourable if G admits a homomorphism to $K_{(n,d)}$. The *circular chromatic number* (also known as the *star chromatic number* [15]) $\chi_c(G)$ of a graph G is the minimum of those ratios n/d for which $\gcd(n, d) = 1$ and G admits a homomorphism to $K_{(n,d)}$ (it can be shown that one may only consider onto-vertex homomorphisms [17]). We denote by $[m]$ the set $\{1, 2, \ldots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all n-subsets of $[m]$. The *Kneser graph* $\mathrm{KG}(m, n)$ has vertex set $\binom{[m]}{n}$, in which A is connected to B if and only if $A \cap B = \emptyset$. It was conjectured by Kneser [8] in 1955 and proved by Lovász [10] in 1978 that $\chi(\mathrm{KG}(m, n)) = m - 2n + 2$. The *fractional chromatic number*, $\chi_f(G)$, of a graph G is defined as

$$\chi_f(G) \stackrel{\text{def}}{=} \inf\left\{ \frac{m}{n} \,\middle|\, \mathrm{Hom}(G, \mathrm{KG}(m, n)) \neq \emptyset \right\}.$$

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(For more about this parameter see [13].) Given two graphs G and K, we define

$$\gamma(G, K) \stackrel{\text{def}}{=} \max_{H \leqslant G} \left\{ \frac{|E(H)|}{|V(H)|} \middle| \operatorname{Hom}(H, K) \neq \emptyset \right\}$$

as the maximum density of the subgraphs that are homomorphic to H. In the following theorem we show that the parameter γ can be used in a no-homomorphism criterion (for other related results see [1–4,12]).

Theorem 1. *Let G and H be two graphs such that the automorphism group of H acts transitively both on $V(H)$ and $E(H)$ and assume that* $\operatorname{Hom}[G, H] \neq \emptyset$. *Then for any graph K we have*

$$\frac{|V(G)|\gamma(G, K)}{|E(G)|} \geqslant \frac{|V(H)|\gamma(H, K)}{|E(H)|}.$$

Proof. Let $\sigma \in \operatorname{Hom}[G, H]$, $|\operatorname{Aut}(H)| = t$, $\operatorname{Aut}(H) = \{\zeta_i | i = 1, \ldots, t\}$ and define

$$\tilde{G} \stackrel{\text{def}}{=} \bigcup_{i=1}^{t} G_i,$$

where each component of $\tilde{G}$, such as G_i $(i = 1, \ldots, t)$, is an isomorphic copy of G. Also, define the homomorphism $\tilde{\sigma}$ such that its restriction to G_i is $\zeta_i \circ \sigma$. It is easy to see that $\tilde{\sigma} \in \operatorname{Hom}^e[G, H]$, and that for any $e \in E(H)$ and any $v \in V(H)$ we have

$$|\tilde{\sigma}^{-1}(e)| = \frac{|E(G)|}{|E(H)|} \times t \quad \text{and} \quad |\tilde{\sigma}^{-1}(v)| = \frac{|V(G)|}{|V(H)|} \times t.$$

Let $H_0 \leqslant H$ be an induced subgraph of H such that $\gamma(H, K) = |E(H_0)|/|V(H_0)|$, and define $G'_i \stackrel{\text{def}}{=} G_i \cap \tilde{\sigma}^{-1}(H_0)$ $(i = 1, \ldots, t)$. Now,

$$\frac{(t \times |E(G)|/|E(H)|) \times |E(H_0)|}{(t \times |V(G)|/|V(H)|) \times |V(H_0)|} = \frac{|E(G'_1)| + |E(G'_2)| + \cdots + |E(G'_t)|}{|V(G'_1)| + |V(G'_2)| + \cdots + |V(G'_t)|} \leqslant \gamma(G, K),$$

and the theorem follows. □

Note that one can also define similar parameters and prove variants of this theorem (e.g. one can consider all (not necessarily induced) subgraphs). As one of the applications we mention the following corollary about the fractional chromatic number of graphs.

Corollary 1. *For any graph G we have*, $\chi_f(G) \geqslant (|E(G)|/|V(G)|\gamma(G, K_2)) + 1$.

Proof. Let the fractional chromatic number of G be m/n, consider the Kneser graph $H \stackrel{\text{def}}{=} \operatorname{KG}(m, n)$, and define,

$$A \stackrel{\text{def}}{=} \{X \subseteq [m] \,|\, |X| = n, 1 \in X, 2 \notin X\},$$
$$B \stackrel{\text{def}}{=} \{X \subseteq [m] \,|\, |X| = n, 1 \notin X, 2 \in X\}.$$

Let H_0 be the induced (bipartite) graph on $A \cup B$ in $KG(m, n)$. We note that $|E(H_0)|/|V(H_0)| \leqslant \gamma(\operatorname{KG}(m, n), K_2)$, and consequently, by Theorem 1 we have

$$\frac{|V(H)| \times |E(H_0)|}{|E(H)| \times |V(H_0)|} = \frac{\binom{m-n-1}{n-1}}{\binom{m-n}{n}} = \frac{1}{\chi_f(G) - 1} \leqslant \frac{|V(G)|\gamma(G, K_2)}{|E(G)|},$$

which gives the desired lower bound for the fractional chromatic number. □

In what follows we consider one more no-homomorphism criterion, which, although simply proved, has some interesting consequences. For any graph G, let $G^{(l)}$ be the lth power of G, that is obtained on the vertex set $V(G)$, by connecting any two vertices u and v, if there exists a walk of length l between u and v in G. Note that lth power of a simple graph is not necessarily a simple graph itself (e.g. when l is an even integer then the lth power may have loops on its vertices). The following lemma is an easy, but in our opinion effective, observation (the case $l = 3$ has also been used independently by Tardif [14]. Also, for some other applications of density and power graphs in homomorphism problem see [12]).

Lemma 1. *For any two simple graphs G and H and any positive integer l,*

$$\mathrm{Hom}[G, H] \neq \emptyset \quad \Rightarrow \quad \mathrm{Hom}[G^{(l)}, H^{(l)}] \neq \emptyset.$$

Proof. Note that for any positive integer l, homomorphisms preserve walks of length l. $\square$

It must be noted that the lemma is trivially true when $H^{(l)}$ contains a loop (e.g. when $l = 2$). Also, one may deduce the following corollary when $H^{(l)}$ is a simple graph.

Corollary 2. *For any positive integer l and any simple graph G, if $G^{(l)}$ is a core then G is a core.*

Proof. The corollary follows from the fact that if an induced homomorphism $\tilde{\sigma} : G^{(l)} \longrightarrow G^{(l)}$ is an automorphism the $\sigma : G \longrightarrow G$ itself must be an automorphism too. $\square$

As one more immediate consequence of Lemma 1 we may refer to the following corollary that seems to provide a simple proof compared to other well-known results [4,17].

Corollary 3. *Let P and C be the Petersen and the Coxeter graphs, respectively.*

(a) *For any vertex $v \in V(P)$, $\chi_c(P - v) = 3$.*
(b) $\mathrm{Hom}[C, C_7] = \emptyset$.

Proof. It is easy to check that for any vertex $v \in V(P)$, $(P - v)^{(3)} = K_9$. Also, we note that C_5 and $K_{(8,3)}$ do not contain triangles, and consequently, the third power of these graphs are K_5 and K_8, respectively. Therefore, (a) follows from Lemma 1, and on the other hand, (b) can be proved similarly since it is easy to check that $C^{(5)} = K_{28}$. $\square$

As one more observation we consider the following conjecture of Nešetřil.

Conjecture 1 (*Nešetřil's Pentagon Conjecture [11]*). *If G is a cubic graph of sufficiently large girth then* $\mathrm{Hom}[G, C_5] \neq \emptyset$.

It should be noted that the conjecture would be true as a consequence of Brook's theorem, if one would replace C_5 by C_3. On the other hand, the conjecture is known to be false if one replaces C_5 by C_{11}, C_9 or C_7 [6,9,16].

If the answer to Conjecture 1 is *yes* then by Lemma 1 one may deduce that the there exists a number g_0 such that the chromatic number of the third power of any cubic graph whose girth is larger than g_0 is less than six. Hence, we believe that the following problem deserves a closer inspection.

Problem 1. *Is it true that for any natural number g_0 there exists a cubic graph G whose girth is larger that g_0 and* $\chi(G^{(3)}) \geqslant 6$.

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References

[1] J.A. Bondy, P. Hell, A note on the star chromatic number, J. Graph Theory 14 (1990) 479–482.
[2] A. Daneshgar, H. Hajiabolhassan, Graph homomorphims through random walks, J. Graph Theory 44 (2003) 15–38.
[3] A. Daneshgar, H. Hajiabolhassan, Random walks and graph homomorphisms, in: J. Nesetril, P. Winkler (Eds.), Graph Morphisms and Statistical Physics, AMS Proceedings DIMACS Series, vol. 63, 2004, pp. 49–63.
[4] A. Daneshgar, H. Hajiabolhassan, Circular colouring and algebraic no-homomorphism theorems, European J. Combin. 28 (2007) 1843–1853.
[5] G. Hahn, C. Tardif, Graph homomorphisms: structure and symmetry, in: G. Hahn, G. Sabidussi (Eds.), Graph Symmetry, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, vol. 497, Kluwer Academic Publishers, Dordrecht, 1997, pp. 107–167.
[6] H. Hatami, Random cubic graphs are not homomorphic to the cycle of size 7, J. Combin. Theory Ser. B 93 (2005) 319–325.
[7] P. Hell, J. Nešetřil, Graphs and Homomorphisms, Oxford Lecture Series in Mathematics and its Applications, vol. 28, Oxford University Press, Oxford, 2004.
[8] M. Kneser, Aufgabe 300, Jber. Deutsch. Math.-Verein. 58 (1955) 27.
[9] A. Kostochka, J. Nesetril, P. Smolikova, Colorings and homomorphisms of degenerate and bounded degree graphs, Discrete Math. 233 (2001) 257–266.
[10] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, J. Combin. Theory Ser. A 25 (1978) 319–324.
[11] J. Nesetril, Aspects of structural combinatorics (graph homomorphisms and their use), Taiwanese J. Math. 3 (1999) 381–423.
[12] J. Nesetril, P. Ossona De Mendez, Colorings and homomorphisms of minor closed classes, in: B. Aronov, S. Basu, J. Pach, M. Sharir (Eds.), Discrete and Computational Geometry: The Goodman–Pollack Festschrift, Springer, Berlin, 2003, pp. 651–664.
[13] E.R. Scheinerman, D.H. Ullman, Fractional Graph Theory, Wiley, New York, 1997.
[14] C. Tardif, Multiplicative graphs and semi-lattice endomorphisms in the category of graphs, J. Combin. Theory Ser. B 95 (2005) 338–345.
[15] A. Vince, Star chromatic number, J. Graph Theory 12 (1988) 551–559.
[16] I.M. Wanless, N.C. Wormald, Regular graphs with no homomorphisms onto cycles, J. Combin. Theory Ser. B 82 (2001) 155–160.
[17] X. Zhu, Circular chromatic number: a survey, Discrete Math. 229 (2001) 371–410.